

Random Walks and Subfractional Brownian Motion

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Abstract

In this article, we show a result of approximation in law to subfractional Brownian motion, with $H > \frac{1}{2}$, in the Skorohod topology. The construction of these approximations is based on a sequence of I.I.D random variables.

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1. Introduction

Recently, the long-range dependence property has become an important aspect of stochastic models in various scientific areas, such as hydrology, telecommunication, finance and so on. The best known and widely used process that exhibits the long-range dependence property is the fractional Brownian motion (FBM) introduced by Mandelbrot and Van Ness [9]. FBM is a suitable generalization of the standard Brownian motion. It exhibits long-range dependence and self-similarity. Moreover, it has stationary increments. Refer to Samorodnitsky and Taqqu [12] for more information on FBM.

On the other hand, many authors have proposed to use more general self-similar Gaussian processes and random fields as stochastic models. Such models have raised many interesting theoretical questions. However, contrast to the extensive studies on FBM, there has been little systematic investigation on other self-similar processes. It seems to me that the main reason for this is the complexity of dependence structures for self-similar processes, which do not have stationary increments. Therefore, it seems interesting to study these processes.

Recently, Bojdecki et al. [3] introduced and studied a rather special class of self-similar processes which preserves many properties of FBM. This process arises from occupation time fluctuations of branching particle systems with Poisson initial condition. This process

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is called the sub-fractional Brownian motion (Sub-FBM). It is also independently appeared in Dzharidze and Van Zanten [5].

Weak convergence to the FBM processes has been studied extensively since the works of Davydov [4] and Taqqu [14]. In recent years, many new results on approximations of FBM have been established. See, for example, Meyer, Sellan and Taqqu [10], Li and Dai [8] and the references therein. When we consider the Sub-FBM process, a natural question is to ask what processes can approximate to it. In recent years, some results on approximations of Sub-FBMs have been obtained. For example, Bardina and Bascompte [1] presented a weak theorem for Sub-FBM based on a Poisson process. Harnett and Nualart [7] proved weak convergence to Sub-FBM with $H = \frac{1}{6}$. Garzòn et al. [6] proved a strong uniform approximation with a rate of convergence for Sub-FBM by means of transport processes.

Inspired by these results, we will present a approximation to Sub-FBM with $H > \frac{1}{2}$. The paper is organized as follows. In Section 2, we recall some preliminaries for Sub-FBM and present the main result of this paper. Section 3 provides the proof of the main result.

2. Preliminaries and Main result

In this section, we first briefly recall some basic preliminaries for Sub-FBM. The sub-fractional Brownian motion $X^H = \{X^H(t), t \geq 0\}$ with index $H \in (0, 1)$ is a centered Gaussian process with covariance function

$$\mathbb{E}[X^H(t)X^H(s)] = s^{2H} + t^{2H} - \frac{1}{2}[(s+t)^{2H} + |t-s|^{2H}]. \quad (2.1)$$

Furthermore, it satisfies the following estimates for $s < t$

$$[(2 - 2^{2H-1}) \wedge 1](t-s)^{2H} \leq \mathbb{E}[(X^H(t) - X^H(s))^2] \leq [(2 - 2^{2H-1}) \vee 1]^{2H}(t-s)^{2H} \quad (2.2)$$

Sub-FBM X^H is neither a semimartingale nor a Markov process unless $H = \frac{1}{2}$. When $H = \frac{1}{2}$, we have the standard Brownian motion W . Sub-FBM has properties analogous to those of FBM (self-similarity, long-range dependence and Hölder paths). But its increments are not stationary. More works on Sub-FBM can be found in Tudor [15], Yan and Shen [17, 18] and the references therein.

In this paper, we assume the index $H \in (\frac{1}{2}, 1)$. Using the Hankel transform in Dzharidze and Van Zanten [5], we get from Tudor [16] that the process

$$W(t) = \int_0^t \phi_H(t, s) dX^H(s) \quad (2.3)$$

is the unique Brownian motion such that

$$X^H(t) = \int_0^t K_H(t, s) W(ds), \quad t \in [0, 1], \quad (2.4)$$

where

$$\phi_H(t, s) = \frac{s^{H-\frac{1}{2}}}{\Gamma(\frac{3}{2}-H)} \left[t^{H-\frac{3}{2}}(t^2-s^2)^{\frac{1}{2}-H} - (H-\frac{3}{2}) \int_s^t (x^2-s^2)^{\frac{1}{2}-H} x^{H-\frac{3}{2}} dx \right] 1_{(0, t)}(s),$$

and

$$K_H(t, s) = \frac{C_H \sqrt{\pi}}{2^{H-1} \Gamma(H - \frac{1}{2})} s^{\frac{3}{2}-H} \int_s^t (x^2 - s^2)^{H-\frac{3}{2}} dx 1_{(0,t)}(s),$$

with C_H being the normalizing constant. Moreover, X^H and W generate the same filtration.

Let us recall some known facts. Consider a sequence of I.I.D random variables $\{\xi_i\}_{i \in \mathbb{N}}$ with $\mathbb{E}[\xi_i] = 0$ and $\mathbb{E}[\xi_i^2] = 1$. The Donkser invariance principle states that the sequence of the processes

$$W_n(t) = \sum_{k=1}^{\lfloor nt \rfloor} \xi_k \quad (2.5)$$

converges weakly to a standard Brownian motion in the Skorohod topology. Here $\lfloor x \rfloor$ stands for the greatest integer not exceeding x .

This result has been extended by Sottinen [13] to the fractional Brownian motion. Define

$$F_n(t, s) = n \int_{s-\frac{1}{n}}^s F\left(\frac{\lfloor nt \rfloor}{n}, u\right) du, \quad n \geq 1 \quad (2.6)$$

where F is the kernel that transforms the standard Brownian motion into a fractional Brownian one, i.e.,

$$F(t, s) = c_H \left(H - \frac{1}{2}\right) s^{\frac{1}{2}-H} \int_s^t u^{H-\frac{1}{2}} (u-s)^{H-\frac{3}{2}} du,$$

where c_H is still the normalizing constant.

Set

$$Z_n(t) = \int_0^t F_n(t, s) W_n(ds) = \sum_{i=1}^{\lfloor nt \rfloor} n \int_{\frac{i-1}{n}}^{\frac{i}{n}} F\left(\frac{\lfloor nt \rfloor}{n}, u\right) du \frac{\xi_i}{\sqrt{n}}. \quad (2.7)$$

Then $\{Z_n(t)\}$ converges weakly to FBM (see Sottinen [13]).

Inspired by Sottinen [13], we define:

$$K_n(t, s) = n \int_{s-\frac{1}{n}}^s K_H\left(\frac{\lfloor nt \rfloor}{n}, u\right) du, \quad n \geq 1. \quad (2.8)$$

Then $K_n(t, \cdot)$ is an approximation of $K_H(t, \cdot)$ for every $t \in [0, 1]$

Set

$$X_n(t) = n \int_0^t K_n(t, u) W_n(du) = \sum_{k=1}^{\lfloor nt \rfloor} n \int_{\frac{k-1}{n}}^{\frac{k}{n}} K_H\left(\frac{\lfloor nt \rfloor}{n}, u\right) du \frac{\xi_k}{\sqrt{n}}. \quad (2.9)$$

In this paper, we will prove

Theorem 2.1 *The laws of processes $\{X_n(t), t \in [0, 1]\}$ converge weakly, as n tends to infinity, to the law of the sub-fractional Brownian motion X^H given by (2.4).*

In the rest of this paper, most of estimates contain unspecified constants. An unspecified positive and finite constant will be denoted by C , which may not be the same in each occurrence. Sometimes we shall emphasize the dependence of these constants upon parameters.

3. Proof of Theorem 2.1

In this section, we will prove Theorem 2.1. Here, we will verify weak convergence via the convergence of finite-dimensional distributions and tightness.

We first verify the convergence of finite-dimensional distributions.

Lemma 3.1 *The family of stochastic processes $\{X_n(t), t \in [0, 1]\}$ converges in the sense of finite-dimensional distributions to the process X^H defined by (2.4).*

Proof: In order to prove Lemma 3.1, we first establish some facts.

Let us consider an arbitrary sequence of partitions of the interval $[0, 1]$ of the form

$$\pi^m : 0 = t_0 < t_1^m < t_2^m < \cdots < t_m^m = 1,$$

with $|\pi^m| \rightarrow 0$, uniformly as $m \rightarrow \infty$. Define

$$X_m(t) = \sum_{k=1}^m K_H(t, t_{i-1}^m) W(\Delta_i), \quad (3.1)$$

where $\Delta_i = [t_{i-1}^m, t_i^m)$ and

$$W(\Delta_i) = W(t_i^m) - W(t_{i-1}^m).$$

Fact 1.

$$X_m(t) \xrightarrow{L^2} X^H(t), \quad (3.2)$$

as $|\pi^m| \rightarrow 0$, where $\xrightarrow{L^2}$ denotes convergence in $L^2(\Omega)$.

Since $K_H(t, s)$ is continuous and monotone in s for every t , we can easily get that **Fact 1** holds.

Define:

$$X_{m,n}(t) = \sum_{i=1}^m K_H(t, t_{i-1}^m) W_n(\Delta_i), \quad (3.3)$$

where

$$W_n(\Delta_i) = W_n(t_i) - W_n(t_{i-1}).$$

Fact 2.

$$X_{m,n}(t) \xrightarrow{W} X_m(t), \quad (3.4)$$

as $n \rightarrow \infty$, where \xrightarrow{W} denotes weak convergence.

From the invariance principle and the continuous mapping theorem (see e.g., Billingsley [2]), one can easily get that Fact 2 holds.

Set

$$\begin{aligned} \tilde{X}_n(t) &= \sum_{i=1}^{\lfloor nt \rfloor} n \int_{\frac{i-1}{n}}^{\frac{i}{n}} K_H(t, u) du \frac{\xi_i}{\sqrt{n}}; \\ \tilde{X}_{m,n}(t) &= \sum_{k=1}^{\lfloor nt \rfloor} n \int_{\frac{k-1}{n}}^{\frac{k}{n}} \sum_{i=1}^m K_H(t, t_{i-1}^m) 1_{\Delta_i}(u) du \frac{\xi_k}{\sqrt{n}}. \end{aligned}$$

Furthermore, set

$$K^m(t, s) = \sum_{i=1}^m K_H(t, t_{i-1}^m) 1_{\Delta_i}(s). \quad (3.5)$$

Then $K^m(t, \cdot)$ converges to $K(t, \cdot)$ in $L^2[0, 1]$.

Fact 3. As $m \rightarrow \infty$,

$$\tilde{X}_{m,n}(t) \xrightarrow{L^2} \tilde{X}_n(t). \quad (3.6)$$

Indeed, by $\mathbb{E}[\xi_i \xi_j] = 0$ if $i \neq j$, $E[\xi_i^2] = 1$ and the Cauchy-Schwartz inequality, we obtain

$$\begin{aligned} \mathbb{E} \left[\tilde{X}_{m,n}(t) - \tilde{X}_n(t) \right]^2 &= \mathbb{E} \left[\sum_{k=1}^{\lfloor nt \rfloor} \left(n \int_{\frac{k-1}{n}}^{\frac{k}{n}} (K^m(t, u) - K_H(t, u)) du \right) \frac{\xi_k}{\sqrt{n}} \right]^2 \\ &= \sum_{k=1}^{\lfloor nt \rfloor} n \left(\int_{\frac{k-1}{n}}^{\frac{k}{n}} (K^m(t, u) - K_H(t, u)) du \right)^2 \\ &\leq \sum_{k=1}^{\lfloor nt \rfloor} \int_{\frac{k-1}{n}}^{\frac{k}{n}} (K^m(t, u) - K_H(t, u))^2 du \\ &\leq \int_0^1 (K^m(t, u) - K_H(t, u))^2 du \rightarrow 0, \end{aligned}$$

as $m \rightarrow \infty$.

Fact 4. For fixed n and t

$$\tilde{X}_{m,n}(t) - X_{m,n}(t) \xrightarrow{L^2} 0, \quad (3.7)$$

as $m \rightarrow \infty$.

Let λ is the Lebesgue measure. Noting that $\int_{\frac{i-1}{n}}^{\frac{i}{n}} 1_{\Delta_k}(u) du = \lambda([\frac{i-1}{n}, \frac{i}{n}) \cap \Delta_k)$, we can rewrite $\tilde{X}_{m,n}$ as follows,

$$\tilde{X}_{m,n}(t) = \sum_{i=1}^m K_H(t, t_{i-1}^m) \sum_{k=1}^{\lfloor nt \rfloor} n \lambda \left(\left[\frac{k-1}{n}, \frac{k}{n} \right) \cap \Delta_i \right) \frac{\xi_k}{\sqrt{n}}. \quad (3.8)$$

By (3.3), we can get

$$X_{m,n}(t) = \sum_{i=1}^m K_H(t, t_{i-1}^m) \sum_{k=\lfloor nt_{i-1} \rfloor + 1}^{\lfloor nt_i \rfloor} \frac{\xi_k}{\sqrt{n}}. \quad (3.9)$$

We first assume that

Assumption: There exist $\{j_1, j_2, \dots, j_M\} \subseteq \{1, \dots, m\}$ and $\{k_1, k_2, \dots, k_T\} \subseteq \{1, 2, \dots, \lfloor nt \rfloor\}$ such that $[\frac{k_i-1}{n}, \frac{k_i}{n})$ does not belong to Δ_{j_i} and

$$[\frac{k_i-1}{n}, \frac{k_i}{n}) \cap \Delta_{j_i} \neq \emptyset.$$

Noting that if $[\frac{k-1}{n}, \frac{k}{n}] \subseteq \Delta_i$, $\lambda([\frac{k-1}{n}, \frac{k}{n}] \cap \Delta_i) = \frac{1}{n}$, we get

$$\begin{aligned} \mathbb{E} \left[\tilde{X}_{m,n}(t) - X_{m,n}(t) \right]^2 &= \mathbb{E} \left[\sum_{q=1}^M \left(K_H(t, t_{j_q}) - K_H(t, t_{j_{q-1}}) \right) n \sum_{i=1}^T \lambda([k_i - 1, k_i] \cap \Delta_{j_q}) \frac{\xi_{k_i}}{\sqrt{n}} \right]^2 \\ &= \mathbb{E} \left[\sum_{i=1}^T \sum_{q=1}^M \left(K_H(t, t_{j_q}) - K_H(t, t_{j_{q-1}}) \right) \lambda([k_i - 1, k_i] \cap \Delta_{j_q}) \frac{\xi_{k_i}}{\sqrt{n}} \right]^2 \\ &= \sum_{i=1}^T \left[n \int_{\frac{k_i-1}{n}}^{\frac{k_i}{n}} \sum_{q=1}^M \left(K_H(t, t_{j_q}) - K_H(t, t_{j_{q-1}}) \right) 1_{\Delta_{j_q}}(u) du \right]^2 \quad (3.10) \end{aligned}$$

From the assumption, it is easy to find a partition of $[0, 1]$ with $(t_{j_q})_{q=1, \dots, M}$, i.e.,

$$0 = t_0 < t_{j_1} < t_{j_2} < \dots < t_{j_M} \leq t_{j_{M+1}} = t_{j_{M+2}} \dots = t_{j_m} = 1,$$

and $|t_{j_q} - t_{j_{q-1}}| < |t_i - t_{i-1}|$ for some i .

Then, we get that as $m \rightarrow \infty$,

$$|\tilde{\Delta}_{j_q}| = |t_{j_q} - t_{j_{q-1}}| \rightarrow 0.$$

Since $K_H(t, \cdot)$ is continuous and monotonic, we have that as $m \rightarrow \infty$,

$$\begin{aligned} \mathbb{E} \left[\tilde{X}_{m,n}(t) - X_{m,n}(t) \right]^2 &\leq \sum_{i=1}^{\lfloor nt \rfloor} \int_{\frac{k_i-1}{n}}^{\frac{k_i}{n}} \left[\sum_{q=1}^M \left(K_H(t, j_q) - K_H(t, j_{q-1}) \right) 1_{\tilde{\Delta}_{j_q}}(u) du \right]^2 \\ &\leq C \int_0^1 \left[\sum_{q=1}^M K_H(t, j_q) 1_{\tilde{\Delta}_{j_q}}(u) - H_K(t, u) \right]^2 du \rightarrow 0. \quad (3.11) \end{aligned}$$

By (3.11), we get that the convergence is uniformly in n .

On the other hand, if Assumption does not hold, then one can easily get that

$$X_{m,n}(t) = \tilde{X}_{m,n}(t).$$

So Fact 4 holds.

Fact 5. *The family of processes $\{\tilde{X}_n\}$ converges to the process X^H in the sense of finite-dimensional distributions.*

In order to prove **Fact 5**, it suffices to show that for any $t_1, \dots, t_p \in [0, 1]$ and $x \in \mathbb{R}$, we have

$$\mathbb{E} \left[\exp \left(ix \sum_{i=1}^p \tilde{X}_n(t_i) \right) \right] \rightarrow \mathbb{E} \left[\exp \left(ix \sum_{i=1}^p X^H(t_i) \right) \right], \quad (3.12)$$

as $n \rightarrow \infty$.

Note that

$$\left| \mathbb{E} \left[\exp \left(ix \sum_{i=1}^p \tilde{X}_n(t_i) \right) \right] - \mathbb{E} \left[\exp \left(ix \sum_{i=1}^p X^H(t_i) \right) \right] \right| \leq I_1 + I_2 + I_3, \quad (3.13)$$

where

$$\begin{aligned} I_1 &= \left| \mathbb{E} \left[\exp \left(ix \sum_{i=1}^p X_{m,n}(t_i) \right) - \exp \left(ix \sum_{i=1}^p X_m(t_i) \right) \right] \right|, \\ I_2 &= \left| \mathbb{E} \left[\exp \left(ix \sum_{i=1}^p X_{m,n}(t_i) \right) - \exp \left(ix \sum_{i=1}^p \tilde{X}_n(t_i) \right) \right] \right|, \\ I_3 &= \left| \mathbb{E} \left[\exp \left(ix \sum_{i=1}^p X_m(t_i) \right) - \exp \left(ix \sum_{i=1}^p X^H(t_i) \right) \right] \right|. \end{aligned}$$

We first deal with I_2 . By the proof of **Facts 3** and **4**, we have that for fixed $\epsilon > 0$, the terms I_2 can be bounded (uniformly in n) by ϵ when m is large enough. Similarly, we can get that the same result for I_3 . For I_1 , from **Fact 2**, we can get that it goes to zero as $n \rightarrow \infty$. From above arguments, we get that (3.12) holds.

Fact 6.

$$\mathbb{E} \left[X_n(t) - \tilde{X}_n(t) \right]^2 \rightarrow 0, \quad (3.14)$$

as $n \rightarrow \infty$.

Indeed,

$$\begin{aligned} \mathbb{E} \left[\tilde{X}_n(t) - X_n(t) \right]^2 &= \mathbb{E} \left[\sum_{k=1}^{\lfloor nt \rfloor} n \int_{\frac{k-1}{n}}^{\frac{k}{n}} (K_H(t, u) - K_H(\frac{\lfloor nt \rfloor}{n}, u)) du \frac{\xi_k}{\sqrt{n}} \right]^2 \\ &= \sum_{k=1}^{\lfloor nt \rfloor} n \left[\int_{\frac{k-1}{n}}^{\frac{k}{n}} (K_H(t, u) - K_H(\frac{\lfloor nt \rfloor}{n}, u)) du \right]^2 \\ &\leq \int_0^1 (K_H(t, u) - K_H(\frac{\lfloor nt \rfloor}{n}, u))^2 du \rightarrow 0, \end{aligned}$$

since $K_H(t, \cdot)$ is a continuous function.

Next we will use **Facts 1** to **6** to prove **Lemma 3.1**.

In order to prove **Lemma 3.1**, it suffices to prove that for any $t_1, \dots, t_p \in [0, 1]$ and $\eta \in \mathbb{R}$, we have

$$\mathbb{E} \left[\exp \left(i\eta \sum_{i=1}^p X_n(t_i) \right) \right] - \mathbb{E} \left[\exp \left(i\eta \sum_{i=1}^p X^H(t_i) \right) \right] \rightarrow 0, \quad (3.15)$$

as $n \rightarrow \infty$.

On the other hand, we have

$$\left| \mathbb{E} \left[\exp \left(i\eta \sum_{i=1}^p X_n(t_i) \right) - \exp \left(i\eta \sum_{i=1}^p X^H(t_i) \right) \right] \right| \leq D_1 + D_2 + D_3 + D_4 + D_5, \quad (3.16)$$

where

$$\begin{aligned}
D_1 &= \mathbb{E} \left[\exp \left(i\eta \sum_{i=1}^p X_n(t_i) \right) - \exp \left(i\eta \sum_{i=1}^p \tilde{X}_n(t_i) \right) \right]; \\
D_2 &= \mathbb{E} \left[\exp \left(i\eta \sum_{i=1}^p \tilde{X}_{m,n}(t_i) \right) - \exp \left(i\eta \sum_{i=1}^p \tilde{X}_n(t_i) \right) \right]; \\
D_3 &= \mathbb{E} \left[\exp \left(i\eta \sum_{i=1}^p \tilde{X}_{m,n}(t_i) \right) - \exp \left(i\eta \sum_{i=1}^p X_{m,n}(t_i) \right) \right]; \\
D_4 &= \mathbb{E} \left[\exp \left(i\eta \sum_{i=1}^p X_{m,n}(t_i) \right) - \exp \left(i\eta \sum_{i=1}^p X_m(t_i) \right) \right];
\end{aligned}$$

and

$$D_5 = \mathbb{E} \left[\exp \left(i\eta \sum_{i=1}^p X_m(t_i) \right) - \exp \left(i\eta \sum_{i=1}^p X^H(t_i) \right) \right].$$

Then, using the same method as the proof of Fact 4, we can prove (3.15) holds. \square

Next, we prove the tightness of $\{X_n(t)\}_{n \in \mathbb{N}}$.

Lemma 3.2 *The family $\{X_n(t); t \in [0, 1]\}$ given by (2.7) is tight.*

Proof: In order to prove this lemma, it suffices to prove that for any $s < t < r \in [0, 1]$,

$$\mathbb{E} \left[|X_n(t) - X_n(s)| |X_n(r) - X_n(t)| \right] \leq C|r - s|^{2H}. \quad (3.17)$$

since $H > \frac{1}{2}$.

Noting that the kernel $K_H(t, u)$ vanishes when u is larger than t , we have

$$\begin{aligned}
\mathbb{E} \left[X_n(t) - X_n(s) \right]^2 &= \mathbb{E} \left[\sum_{i=1}^{\lfloor nt \rfloor} n \int_{\frac{i-1}{n}}^{\frac{i}{n}} \left(K_H\left(\frac{\lfloor nt \rfloor}{n}, u\right) - K_H\left(\frac{\lfloor ns \rfloor}{n}, u\right) \right) du \frac{\xi_i}{\sqrt{n}} \right]^2 \\
&= \sum_{i=1}^{\lfloor nt \rfloor} n \left(\int_{\frac{i-1}{n}}^{\frac{i}{n}} \left(K_H\left(\frac{\lfloor nt \rfloor}{n}, u\right) - K_H\left(\frac{\lfloor ns \rfloor}{n}, u\right) \right) du \right)^2, \quad (3.18)
\end{aligned}$$

since $\mathbb{E}[\xi_i \xi_j] = 0$ if $i \neq j$, and $\mathbb{E}[\xi_i^2] = 1$.

By the Hölder inequality, we get that (3.18) can be bounded by

$$\begin{aligned}
\sum_{i=1}^{\lfloor nt \rfloor} \int_{\frac{i-1}{n}}^{\frac{i}{n}} \left(K_H\left(\frac{\lfloor nt \rfloor}{n}, u\right) - K_H\left(\frac{\lfloor ns \rfloor}{n}, u\right) \right)^2 du &\leq C \int_0^1 \left(K_H\left(\frac{\lfloor nt \rfloor}{n}, u\right) - K_H\left(\frac{\lfloor ns \rfloor}{n}, u\right) \right)^2 du \\
&\leq C \left| \frac{\lfloor nt \rfloor}{n} - \frac{\lfloor ns \rfloor}{n} \right|^{2H}, \quad (3.19)
\end{aligned}$$

since (2.2) holds.

Therefore, the tightness can be obtained by Billingsley [2, Section 5]. \square

Proof of Theorem 2.1: Theorem 2.1 is a direct consequence of Lemmas 3.1 and 3.2, because tightness plus convergence of finite dimensional distributions implies the weak convergence in the skorohod topology (see Billingsly [2, Section 5]). \square

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References

- [1] Bardina, B., Bascompte, D., 2010. Weak convergence towards two independent Gaussian processes from a unique poisson process. *Collect. Math.* 61, 191-204.
- [2] Billingsley, P., 1968. *Convergence of Probability Measures*. John Wiley and Sons, New York.
- [3] Bojdecki, T., Gorostiza, L.G., Talarczyk, A., 2004. Sub-fractional Brownian motion and its relation to occupation times. *Statist. Probab. Lett.* 69, 405-419.
- [4] Davydov, Y., 1970. The invariance principle for stationary processes. *Teor. Veroyatn. Ee Primenen.* 15, 498-509.
- [5] Dzharidze, K., Van Zanten, H., 2004. A series expansion of fractional Brownian motion. *Probab. Theory Relat. Fields* 103, 39-55.
- [6] Garzón, J., Gorostiza, G., León, A., 2012. A strong uniform approximation of sub-fractional Brownian motion. Preprint.
- [7] Harnett, D., Nualart, D., 2012. Weak convergence of the stratonovich integral with respect to a class of Gaussian processes. Preprint.
- [8] Li, Y., Dai, H., 2011. Approximations of fractional Brownian motion. *Bernoulli* 17, 1195-1216.
- [9] Mandelbrot, B., Van Ness, J.W., 1968. Fractional Brownian motions, fractional noises and applications. *SIAM Rev.* 10, 422-437.
- [10] Meyer, Y., Sellan, F., Taqqu, M.S., 1999. Wavelets, generalized white noise and fractional integration: the synthesis of fractional Brownian motion. *J. Fourier Anal. Appl.* 5, 465-494.
- [11] Nualart, D., 2006. *Malliavin Calculus and related topics (2nd edition)*. Springer, New York.
- [12] Samorodnitsky, G., Taqqu M.S., 1994. *Stable Non-Gaussian Random Processes*. Chapman and Hall, New York.
- [13] Sottinen, T., 2001. Fractional Brownian motion, random walks and binary market models. *Finance and Stochastic* 5, 343-355.
- [14] Taqqu, M.S., 1975. Weak Convergence to fractional Brownian motion and to the Rosenblatt process. *Z. Wahrsch. Verw. Gebiete* 31, 287-302.

- [15] Tudor, C., 2007. Some properties of the sub-fractional Brownian motion. *Stochastics* 79, 431-448.
- [16] Tudor, C., 2009. On the Wiener integral with respect to a sub-fractional Brownian motion on an interval. *J. Math. Anal. Appl.* 351, 456-468.
- [17] Yan, L. Shen, G., 2010. On the collision local time of sub-fractional Brownian motions. *Statist. Probab. Lett.* 80, 296-308.
- [18] Yan, L. Shen, G., He, K., 2011. Itô's formula for a subfractional Brownian motion. *Commun. Stoch. Anal.* 5, 135-159.